Sums of squares, Jacobi forms and differential equations

Alexander Kalmynin*

*International Laboratory of Mirror Symmetry and Automorphic forms NRU HSE, Moscow, Russia

Integrable Systems and Automorphic Forms
Sirius Mathematics Center, Sochi, Russia
February 25, 2020

1. Introduction

Let $\mathcal{S} = \{1, 2, 4, 5, 8, 9, 10, 13, 16 \ldots\} = \{s_1, s_2, s_3, \ldots, s_n, \ldots\}$ be the sequence of all positive integers which are sums of two squares, arranged in ascending order.

1. Introduction

Let $\mathcal{S} = \{1, 2, 4, 5, 8, 9, 10, 13, 16 \ldots\} = \{s_1, s_2, s_3, \ldots, s_n, \ldots\}$ be the sequence of all positive integers which are sums of two squares, arranged in ascending order. It is well-known that $s_n \sim Cn\sqrt{\log n}$ for $C \approx 1.3085$.

1. Introduction

Let $\mathcal{S} = \{1, 2, 4, 5, 8, 9, 10, 13, 16 \ldots\} = \{s_1, s_2, s_3, \ldots, s_n, \ldots\}$ be the sequence of all positive integers which are sums of two squares, arranged in ascending order. It is well-known that $s_n \sim Cn\sqrt{\log n}$ for $C \approx 1.3085$. Next, it is natural to ask about bounds for $s_{n+1} - s_n$.

1. Introduction

Let $\mathcal{S} = \{1, 2, 4, 5, 8, 9, 10, 13, 16 \ldots\} = \{s_1, s_2, s_3, \ldots, s_n, \ldots\}$ be the sequence of all positive integers which are sums of two squares, arranged in ascending order. It is well-known that $s_n \sim Cn\sqrt{\log n}$ for $C \approx 1.3085$. Next, it is natural to ask about bounds for $s_{n+1} - s_n$. The only known upper bound

$$s_{n+1} - s_n \ll s_n^{1/4}$$

easily follows via greedy approximation:

1. Introduction

Let $\mathcal{S} = \{1, 2, 4, 5, 8, 9, 10, 13, 16 \dots\} = \{s_1, s_2, s_3, \dots, s_n, \dots\}$ be the sequence of all positive integers which are sums of two squares, arranged in ascending order. It is well-known that $s_n \sim Cn\sqrt{\log n}$ for $C \approx 1.3085$. Next, it is natural to ask about bounds for $s_{n+1} - s_n$. The only known upper bound

$$s_{n+1} - s_n \ll s_n^{1/4}$$

easily follows via greedy approximation:

$$x = (x - f(x)) + (f(x) - f(f(x))) + f(f(x)),$$

where $f(x) = x - [\sqrt{x}]^2 = O(\sqrt{x})$.

1. Introduction

Let $\mathcal{S} = \{1, 2, 4, 5, 8, 9, 10, 13, 16 \ldots\} = \{s_1, s_2, s_3, \ldots, s_n, \ldots\}$ be the sequence of all positive integers which are sums of two squares, arranged in ascending order. It is well-known that $s_n \sim Cn\sqrt{\log n}$ for $C \approx 1.3085$. Next, it is natural to ask about bounds for $s_{n+1} - s_n$.
The only known upper bound

$$s_{n+1} - s_n \ll s_n^{1/4}$$

easily follows via greedy approximation:

$$x = (x - f(x)) + (f(x) - f(f(x))) + f(f(x)),$$

where $f(x) = x - [\sqrt{x}]^2 = O(\sqrt{x})$.
Some probabilistic models suggest the bound

$$s_{n+1} - s_n \ll (\ln s_n)^{3/2}.$$

1. Introduction

The following lower bounds for $g(N) = \max_{n \leq N}(s_{n+1} - s_n)$ are proved by certain combinatorial considerations:

1. Introduction

The following lower bounds for $g(N) = \max_{n \le N}(s_{n+1} - s_n)$ are proved by certain combinatorial considerations:

- $g(N) \gg \frac{\log N}{\sqrt{\log \log N}}$ (P. Erdős)

1. Introduction

The following lower bounds for $g(N) = \max_{n \le N}(s_{n+1} - s_n)$ are proved by certain combinatorial considerations:

- $g(N) \gg \frac{\log N}{\sqrt{\log \log N}}$ (P. Erdős)
- $g(N) \ge \left(\frac{1}{4} + o(1)\right) \log N$ (I. Richards)

1. Introduction

The following lower bounds for $g(N) = \max_{n \le N}(s_{n+1} - s_n)$ are proved by certain combinatorial considerations:

- $g(N) \gg \frac{\log N}{\sqrt{\log\log N}}$ (P. Erdős)
- $g(N) \ge \left(\frac{1}{4} + o(1)\right)\log N$ (I. Richards)
- $g(N) \ge \left(\frac{195}{449} + o(1)\right)\log N$ (R. Dietmann, C. Elsholtz)

1. Introduction

The following lower bounds for $g(N) = \max_{n \le N}(s_{n+1} - s_n)$ are proved by certain combinatorial considerations:

- $g(N) \gg \frac{\log N}{\sqrt{\log\log N}}$ (P. Erdős)
- $g(N) \ge \left(\frac{1}{4} + o(1)\right)\log N$ (I. Richards)
- $g(N) \ge \left(\frac{195}{449} + o(1)\right)\log N$ (R. Dietmann, C. Elsholtz)
- $g(N) \ge \left(\frac{390}{449} + o(1)\right)\log N$ (A.K., S. Konyagin)

1. Introduction

The following lower bounds for $g(N) = \max_{n \le N}(s_{n+1} - s_n)$ are proved by certain combinatorial considerations:

- $g(N) \gg \frac{\log N}{\sqrt{\log\log N}}$ (P. Erdős)
- $g(N) \ge \left(\frac{1}{4} + o(1)\right) \log N$ (I. Richards)
- $g(N) \ge \left(\frac{195}{449} + o(1)\right) \log N$ (R. Dietmann, C. Elsholtz)
- $g(N) \ge \left(\frac{390}{449} + o(1)\right) \log N$ (A.K., S. Konyagin)

One can also prove bounds for moments of gaps. The first result in this direction is by C. Hooley:

$$\sum_{s_{n+1} \le x} (s_{n+1} - s_n)^\gamma \ll x(\log x)^{0.5(\gamma-1)}$$

for all $\gamma < \frac{5}{3}$.

1. Introduction

The following lower bounds for $g(N) = \max_{n \le N}(s_{n+1} - s_n)$ are proved by certain combinatorial considerations:

- $g(N) \gg \frac{\log N}{\sqrt{\log\log N}}$ (P. Erdős)
- $g(N) \ge \left(\frac{1}{4} + o(1)\right) \log N$ (I. Richards)
- $g(N) \ge \left(\frac{195}{449} + o(1)\right) \log N$ (R. Dietmann, C. Elsholtz)
- $g(N) \ge \left(\frac{390}{449} + o(1)\right) \log N$ (A.K., S. Konyagin)

One can also prove bounds for moments of gaps. The first result in this direction is by C. Hooley:

$$\sum_{s_{n+1} \le x} (s_{n+1} - s_n)^\gamma \ll x(\log x)^{0.5(\gamma-1)}$$

for all $\gamma < \frac{5}{3}$. To prove this, Hooley constructs certain function $f(n)$ with $f(n) = 0$ for $n \notin \mathcal{S}$ and proves upper bounds for the quantity

$$\sum_{n \le x} \left(\sum_{n < m \le n+h} f(m) - \Delta h \right)^2$$

for certain values of h and Δ.

1. Introduction

The following lower bounds for $g(N) = \max_{n \le N}(s_{n+1} - s_n)$ are proved by certain combinatorial considerations:

- $g(N) \gg \frac{\log N}{\sqrt{\log\log N}}$ (P. Erdős)
- $g(N) \ge \left(\frac{1}{4} + o(1)\right)\log N$ (I. Richards)
- $g(N) \ge \left(\frac{195}{449} + o(1)\right)\log N$ (R. Dietmann, C. Elsholtz)
- $g(N) \ge \left(\frac{390}{449} + o(1)\right)\log N$ (A.K., S. Konyagin)

One can also prove bounds for moments of gaps. The first result in this direction is by C. Hooley:

$$\sum_{s_{n+1} \le x} (s_{n+1} - s_n)^\gamma \ll x(\log x)^{0.5(\gamma-1)}$$

for all $\gamma < \frac{5}{3}$. To prove this, Hooley constructs certain function $f(n)$ with $f(n) = 0$ for $n \notin \mathcal{S}$ and proves upper bounds for the quantity

$$\sum_{n \le x} \left(\sum_{n < m \le n+h} f(m) - \Delta h \right)^2$$

for certain values of h and Δ. Proof relies on bounds for Kloosteman sums.

2. Main identity

Now we are going to discuss a (relatively) recent improvement of Hooley's result, which allows us to extend the range of admissible γ to $\gamma < 2$.

2. Main identity

Now we are going to discuss a (relatively) recent improvement of Hooley's result, which allows us to extend the range of admissible γ to $\gamma < 2$.

Theorem 1 (K.,2018)

For any $1 \leq \gamma < 2$ *we have*

$$\sum_{s_{n+1} \leq x} (s_{n+1} - s_n)^{\gamma} \ll x(\log x)^{1.5(\gamma-1)}.$$

2. Main identity

Now we are going to discuss a (relatively) recent improvement of Hooley's result, which allows us to extend the range of admissible γ to $\gamma < 2$.

Theorem 1 (K.,2018)

For any $1 \le \gamma < 2$ *we have*

$$\sum_{s_{n+1} \le x} (s_{n+1} - s_n)^\gamma \ll x(\log x)^{1.5(\gamma-1)}.$$

To prove this, we need to consider the function

$$\Theta(\tau; z) = \sum_{n \ge 0} r_2(n) J_0(2\pi\sqrt{n}z) e^{\pi i n \tau}.$$

Here $z \in \mathbb{C}$, $\tau \in \mathbb{H}$, $J_0(2\sqrt{z}) = \sum \frac{(-1)^n z^n}{n!^2}$ and $r_2(n) = \#\{(a,b) \in \mathbb{Z}^2 : n = a^2 + b^2\}$.

2. Main identity

It turns out that $\Theta(\tau; z)$ behaves like Jacobi form. In particular, it satisfies the following identity

2. Main identity

It turns out that $\Theta(\tau; z)$ behaves like Jacobi form. In particular, it satisfies the following identity

$$\Theta\left(-\frac{1}{\tau}, \frac{z}{\tau}\right) = -i\tau \exp\left(\frac{4\pi i z^2}{\tau}\right)\Theta(\tau, z).$$

2. Main identity

It turns out that $\Theta(\tau; z)$ behaves like Jacobi form. In particular, it satisfies the following identity

$$\Theta\left(-\frac{1}{\tau}, \frac{z}{\tau}\right) = -i\tau \exp\left(\frac{4\pi i z^2}{\tau}\right)\Theta(\tau, z).$$

There are several ways to prove this formula. For example, one can use general theorem by N.V. Kuznetsov:

Theorem 2

Let $f(\tau)$ be a modular form of type (λ, k, w), i.e. $f(\tau+\lambda) = f(\tau)$ and $f\left(-\frac{1}{\tau}\right) = w(-i\tau)^k f(\tau)$. Assume that $f(\tau) = \sum_{n\geq 0} a(n)e^{2\pi i\tau n/\lambda}$.

2. Main identity

It turns out that $\Theta(\tau; z)$ behaves like Jacobi form. In particular, it satisfies the following identity

$$\Theta\left(-\frac{1}{\tau}, \frac{z}{\tau}\right) = -i\tau \exp\left(\frac{4\pi i z^2}{\tau}\right)\Theta(\tau, z).$$

There are several ways to prove this formula. For example, one can use general theorem by N.V. Kuznetsov:

Theorem 2

Let $f(\tau)$ be a modular form of type (λ, k, w), i.e. $f(\tau+\lambda) = f(\tau)$ and $f\left(-\frac{1}{\tau}\right) = w(-i\tau)^k f(\tau)$. Assume that $f(\tau) = \sum_{n\geq 0} a(n)e^{2\pi i\tau n/\lambda}$. Then the function

$$g_f(\tau, z) = \frac{(2\pi)^{k-1}a(0)}{\Gamma(k)} + \sum_{n>0} a(n)e^{2\pi i\tau n/\lambda}\frac{J_{k-1}(4\pi z\sqrt{n})}{(z\sqrt{n})^{k-1}}$$

satisfies $g_f\left(-\frac{1}{\tau}, \frac{z}{\tau}\right) = w(-i\tau)^k \exp\left(\frac{2\pi i z^2\lambda}{\tau}\right) g_f(\tau, z)$.

2. Main identity

One can also prove our identity using Laplace transform of Bessel function or confluent hypergeometric functions and functional equation for the corresponding L-function.

2. Main identity

One can also prove our identity using Laplace transform of Bessel function or confluent hypergeometric functions and functional equation for the corresponding L-function. Other way to do it is to consider the integral

$$I(z,\tau) = \frac{1}{2\pi}\int_0^{2\pi} \vartheta(\tau, z\cos\varphi)\vartheta(\tau, z\sin\varphi)d\varphi,$$

2. Main identity

One can also prove our identity using Laplace transform of Bessel function or confluent hypergeometric functions and functional equation for the corresponding L-function. Other way to do it is to consider the integral

$$I(z,\tau) = \frac{1}{2\pi}\int_0^{2\pi} \vartheta(\tau, z\cos\varphi)\vartheta(\tau, z\sin\varphi)d\varphi,$$

observe that $I(z,\tau) = \Theta(z,\tau)$ and use the fact that $\vartheta(\tau, z)$ is a Jacobi form.

One can also prove our identity using Laplace transform of Bessel function or confluent hypergeometric functions and functional equation for the corresponding L-function. Other way to do it is to consider the integral

$$I(z,\tau)=\frac{1}{2\pi}\int_0^{2\pi}\vartheta(\tau,z\cos\varphi)\vartheta(\tau,z\sin\varphi)d\varphi,$$

observe that $I(z,\tau)=\Theta(z,\tau)$ and use the fact that $\vartheta(\tau,z)$ is a Jacobi form.
To derive Theorem 1 from the main identity, let us choose large N that is far from all sums of two squares and large parameter M. Define

$$S(N,M)=\Theta(iM^{-1},\sqrt{N})=\sum_{n\geq 0}r_2(n)J_0(2\pi\sqrt{nN})e^{-\pi n/M}.$$

2. Main identity

Our transformation formula gives

$$S(N, M) = Me^{-\pi NM} \sum_{n \geq 0} r_2(n) I_0(2\pi M\sqrt{nN}) e^{-\pi nM}.$$

2. Main identity

Our transformation formula gives

$$S(N,M) = Me^{-\pi NM} \sum_{n\geq 0} r_2(n) I_0(2\pi M\sqrt{nN}) e^{-\pi nM}.$$

Now, as $z \to +\infty$ we have $I_0(z) \sim \frac{e^z}{\sqrt{2\pi z}}$, so that

$$Me^{-\pi NM} I_0(2\pi M\sqrt{nN}) e^{-\pi nM} \sim \frac{\sqrt{M} e^{-\pi M(\sqrt{n}-\sqrt{N})^2}}{2\pi \sqrt[4]{nN}}.$$

2. Main identity

Our transformation formula gives

$$S(N,M) = Me^{-\pi NM} \sum_{n\geq 0} r_2(n) I_0(2\pi M\sqrt{nN}) e^{-\pi nM}.$$

Now, as $z \to +\infty$ we have $I_0(z) \sim \frac{e^z}{\sqrt{2\pi z}}$, so that

$$Me^{-\pi NM} I_0(2\pi M\sqrt{nN}) e^{-\pi nM} \sim \frac{\sqrt{M} e^{-\pi M(\sqrt{n}-\sqrt{N})^2}}{2\pi \sqrt[4]{nN}}.$$

From this one can easily derive that if $|N - a^2 - b^2| \geq H$ for all integers a, b and $M = \frac{10N\log N}{H^2}$ then $S(N,M) = o(1)$.

2. Main identity

Our transformation formula gives

$$S(N,M) = Me^{-\pi NM} \sum_{n\geq 0} r_2(n) I_0(2\pi M\sqrt{nN}) e^{-\pi nM}.$$

Now, as $z \to +\infty$ we have $I_0(z) \sim \frac{e^z}{\sqrt{2\pi z}}$, so that

$$Me^{-\pi NM} I_0(2\pi M\sqrt{nN}) e^{-\pi nM} \sim \frac{\sqrt{M} e^{-\pi M(\sqrt{n}-\sqrt{N})^2}}{2\pi \sqrt[4]{nN}}.$$

From this one can easily derive that if $|N - a^2 - b^2| \geq H$ for all integers a, b and $M = \frac{10N \log N}{H^2}$ then $S(N,M) = o(1)$.
On the other hand, the functions $N \mapsto J_0(2\pi\sqrt{nN})$ oscillate with different phases for different nonzero n, which means that they are essentially orthogonal to each other. For more rigorous argument one should use Weber's second exponential integral:

2. Main identity

Our transformation formula gives

$$S(N,M) = Me^{-\pi NM} \sum_{n \geq 0} r_2(n) I_0(2\pi M\sqrt{nN}) e^{-\pi nM}.$$

Now, as $z \to +\infty$ we have $I_0(z) \sim \frac{e^z}{\sqrt{2\pi z}}$, so that

$$Me^{-\pi NM} I_0(2\pi M\sqrt{nN}) e^{-\pi nM} \sim \frac{\sqrt{M} e^{-\pi M(\sqrt{n}-\sqrt{N})^2}}{2\pi \sqrt[4]{nN}}.$$

From this one can easily derive that if $|N - a^2 - b^2| \geq H$ for all integers a, b and $M = \frac{10N \log N}{H^2}$ then $S(N,M) = o(1)$.
On the other hand, the functions $N \mapsto J_0(2\pi\sqrt{nN})$ oscillate with different phases for different nonzero n, which means that they are essentially orthogonal to each other. For more rigorous argument one should use Weber's second exponential integral:

$$\int_0^{+\infty} J_0(2\sqrt{\alpha x}) J_0(2\sqrt{\beta x}) e^{-\gamma x} dx = \frac{1}{\gamma} I_0\left(\frac{2\sqrt{\alpha\beta}}{\gamma}\right) e^{-(\alpha+\beta)/\gamma}.$$

2. Main identity

Using this observation, one can show that

$$\int_0^N (S(x,M)-1)^2 dx \ll \sqrt{NM}\log N$$

Using this observation, one can show that

$$\int_0^N (S(x,M)-1)^2 dx \ll \sqrt{NM}\log N$$

and

$$\mu(x \leq N : \min_{s\in\mathcal{S}} |x-s| \leq H) \ll \frac{N(\log N)^{3/2}}{H}.$$

Using this observation, one can show that

$$\int_0^N (S(x,M)-1)^2 dx \ll \sqrt{NM}\log N$$

and

$$\mu(x \le N : \min_{s\in\mathcal{S}} |x-s| \le H) \ll \frac{N(\log N)^{3/2}}{H}.$$

One of the clear downsides of this method is the fact that $S(x,M)-1$ can be far from zero for values of x that are not so far from the nearest sum of two squares.

3. Differential equations

There is yet another way to understand (at least, qualitatively), why $\Theta(\tau, z)$ is useful for detection of large gaps in $\mathcal{S}$:

3. Differential equations

There is yet another way to understand (at least, qualitatively), why $\Theta(\tau, z)$ is useful for detection of large gaps in $\mathcal{S}$: we have

$$\pi i \frac{\partial \Theta(\tau, \sqrt{z})}{\partial \tau} = \frac{\partial}{\partial z} \left(z \frac{\partial \Theta(\tau, \sqrt{z})}{\partial z} \right),$$

because $J_0(2\pi\sqrt{nz})e^{\pi i n \tau}$ is the standing wave of this equation.

3. Differential equations

There is yet another way to understand (at least, qualitatively), why $\Theta(\tau, z)$ is useful for detection of large gaps in $\mathcal{S}$: we have

$$\pi i \frac{\partial \Theta(\tau, \sqrt{z})}{\partial \tau} = \frac{\partial}{\partial z}\left(z \frac{\partial \Theta(\tau, \sqrt{z})}{\partial z}\right),$$

because $J_0(2\pi\sqrt{nz})e^{\pi i n \tau}$ is the standing wave of this equation.
On the other hand, from Voronoi summation formula we get

$$\Theta(0, \sqrt{z}) = \sum_{n \geq 0} J_0(2\pi\sqrt{nz}) = \frac{1}{\pi}\sum_{n \geq 0} r_2(n)\delta(z - n).$$

3. Differential equations

There is yet another way to understand (at least, qualitatively), why $\Theta(\tau, z)$ is useful for detection of large gaps in $\mathcal{S}$: we have

$$\pi i \frac{\partial \Theta(\tau, \sqrt{z})}{\partial \tau} = \frac{\partial}{\partial z}\left(z \frac{\partial \Theta(\tau, \sqrt{z})}{\partial z}\right),$$

because $J_0(2\pi\sqrt{nz})e^{\pi i n \tau}$ is the standing wave of this equation.
On the other hand, from Voronoi summation formula we get

$$\Theta(0, \sqrt{z}) = \sum_{n \geq 0} J_0(2\pi\sqrt{nz}) = \frac{1}{\pi}\sum_{n \geq 0} r_2(n)\delta(z-n).$$

Of course, we can also try to go a little deeper.
Transformation formula implies that

$$\Theta(\tau, \sqrt{z}) = \exp\left(-\frac{\pi^2}{6} z E_2(\tau)\right)\sum_{n \geq 0} f_n(\tau) z^n$$

for $E_2 = 1 - 24\sum_{n \geq 1}\sigma_1(n)e^{2\pi i n \tau}$ and some modular forms $f_n(\tau)$ of weight $2n+1$

3. Differential equations

It turns out that these modular forms can be described in terms of certain sequence of polynomials satisfying a differential recurrence relation.

3. Differential equations

It turns out that these modular forms can be described in terms of certain sequence of polynomials satisfying a differential recurrence relation. For all n the form f_n/θ^2 is automorphic for the group $\Gamma(2)$. More precisely, let us define "male theta-series" by

$$\theta_M(\tau) = \sum_{n\in\mathbb{Z}} (-1)^n q^{n^2}$$

3. Differential equations

It turns out that these modular forms can be described in terms of certain sequence of polynomials satisfying a differential recurrence relation. For all n the form f_n/θ^2 is automorphic for the group $\Gamma(2)$. More precisely, let us define "male theta-series" by

$$\theta_M(\tau) = \sum_{n\in\mathbb{Z}} (-1)^n q^{n^2}$$

and "female theta-series" by

$$\theta_F(\tau) = \sum_{n\in\mathbb{Z}+1/2} q^{n^2}$$

3. Differential equations

It turns out that these modular forms can be described in terms of certain sequence of polynomials satisfying a differential recurrence relation. For all n the form f_n/θ^2 is automorphic for the group $\Gamma(2)$. More precisely, let us define "male theta-series" by

$$\theta_M(\tau) = \sum_{n\in\mathbb{Z}} (-1)^n q^{n^2}$$

and "female theta-series" by

$$\theta_F(\tau) = \sum_{n\in\mathbb{Z}+1/2} q^{n^2}$$

Define also $X = -\theta_M(\tau/2)^4/6$, $Y = \theta_F(\tau/2)^4/6$ and $u = \frac{Y}{X}$.

3. Differential equations

It turns out that these modular forms can be described in terms of certain sequence of polynomials satisfying a differential recurrence relation. For all n the form f_n/θ^2 is automorphic for the group $\Gamma(2)$. More precisely, let us define "male theta-series" by

$$\theta_M(\tau) = \sum_{n\in\mathbb{Z}} (-1)^n q^{n^2}$$

and "female theta-series" by

$$\theta_F(\tau) = \sum_{n\in\mathbb{Z}+1/2} q^{n^2}$$

Define also $X = -\theta_M(\tau/2)^4/6$, $Y = \theta_F(\tau/2)^4/6$ and $u = \frac{Y}{X}$.
Then $f_n(\tau) = \pi^{2n} X^n \theta^2(\tau) p_n(u) = \pi^{2n} X^n \sqrt{6Y - 6X} p_n(u)$ for some polynomials p_n with rational coefficients.

3. Differential equations

More precisely, we have $p_0(u) = 1$ and for all $n \geq 0$

$$(n+1)p_{n+1}(u) + p_n(u)(u(1-4n) + 2n + 1) + 6(u^2 - u)p_n'(u) +$$
$$+(u^2 - u + 1)p_{n-1}(u) = 0.$$

Here are the first few values of p_n:

$$p_0 = 1, p_1 = -u - 1,$$
$$p_2 = u^2 - \frac{5}{2}u + 1, p_3 = -\frac{4}{3}u^3 + \frac{3}{2}u^2 + \frac{3}{2}u - \frac{4}{3},$$
$$p_4 = \frac{25}{12}u^4 - \frac{19}{6}u^3 + \frac{21}{8}u^2 - \frac{19}{6}u + \frac{25}{12},$$
$$p_5 = -\frac{209}{60}u^5 + \frac{91}{12}u^4 - \frac{463}{120}u^3 - \frac{463}{120}u^2 + \frac{91}{12}u - \frac{209}{60}$$

Thank you for your attention!